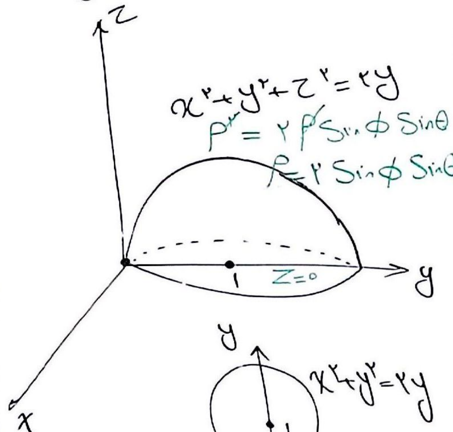
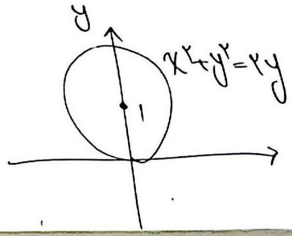
۲. درستی قضیه دیورژانس را برای میدان $\vec{F}=yz\,\vec{i}+z^2\vec{k}$ و S سطح ناحیه $x^2+y^2+z^2\le 2y$ ، $z\geq 0$ با صفحه xy را تحقیق کنید.

$$x^2+y^2+z^2-2y=0$$
$$x^2+y^2-2y+1-1+z^2=0$$
$$x^2+(y-1)^2+z^2=1$$





$$F=\langle yz,\ 0,\ z^2\rangle$$

$$\text{شار}=\iiint M_x+N_y+P_z\,dV=\iiint 2z\,dV \underset{\text{کروی}}{=} \int_0^{\pi}\int_0^{\pi/2}\int_0^{2\sin\phi\sin\theta} \underbrace{2(\rho\cos\phi)\rho^2\sin\phi\,d\rho}_{\frac{2\rho^4}{4}\cos\phi\sin\phi}\,d\phi\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi}\int_0^{\pi/2}\underbrace{(2\sin\phi\sin\theta)^4\cos\phi\sin\phi\,d\phi}_{\frac{8}{3}\sin^4\theta}\,d\theta=\frac{4}{3}\int_0^{\pi}\underbrace{\sin^4\theta}_{\left(\frac{1}{2}(1-\cos 2\theta)\right)^2}\,d\theta$$

$$=\frac{4}{3}\times\frac{1}{4}\int_0^{\pi}1+\underbrace{\cos^2 2\theta}_{\frac{1}{2}(1+\cos 4\theta)}-2\cos 2\theta\,d\theta=\frac{1}{3}\int_0^{\pi}\frac{3}{2}+\frac{1}{2}\cos 4\theta-2\cos 2\theta\,d\theta=\frac{1}{3}\cdot\frac{3}{2}\theta$$

$$=\frac{1}{2}\pi.$$

انتگرال ϕ:

$$16\int_0^{\pi/2}\underbrace{\sin^5\phi}_{t}\,\underbrace{\cos\phi\,d\phi}_{dt}=16\,\frac{\sin^6\phi}{6}=\frac{8}{3}\left(\sin^6\frac{\pi}{2}-\sin^6 0\right)$$

$$=\frac{8}{3}$$

$$\iint F\cdot n\, d\sigma = \iint \left\langle yz,\ 0,\ z^2\right\rangle \cdot \left\langle -\frac{-2x}{2\sqrt{2y-x^2-y^2}},\ -\frac{2-2y}{2\sqrt{2y-x^2-y^2}},\ 1\right\rangle dA = \iint \frac{xyz}{\sqrt{2y-x^2-y^2}} + z^2\, dA$$

$S: z=\sqrt{2y-x^2-y^2}$ تصویر

$$= \iint \frac{xy\sqrt{2y-x^2-y^2}}{\sqrt{2y-x^2-y^2}} + 2y-x^2-y^2\, dA = \iint_{\text{تصویر دایره}} xy+2y-x^2-y^2\, dA \overset{\text{قطبی}}{=} \int_0^{\pi}\int_0^{2\sin\theta} \left((r\cos\theta)(r\sin\theta) + 2r\sin\theta - r^2\right) r\,dr\,d\theta$$

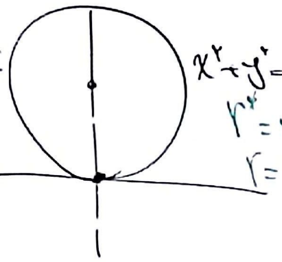
$$\frac{r^4}{4}\sin\theta\cos\theta + \frac{2r^3}{3}\sin\theta - \frac{r^4}{4}$$

$$= \int_0^{\pi} \frac{16\sin^4\theta}{4}\sin\theta\cos\theta + \frac{2}{3}(2\sin\theta)^3\sin\theta - \frac{16\sin^4\theta}{4}\, d\theta$$

$$= \int_0^{\pi} 4\,\underbrace{\sin^5\theta}_{t}\,\cos\theta + \frac{4}{3}\sin^4\theta\, d\theta = 4\,\frac{\sin^6\theta}{6}\Big|_0^{\pi} + \frac{4}{3}\left(\frac{3}{8}\pi\right)$$

$$= 0 + \frac{1}{2}\pi = \frac{1}{2}\pi$$

تصویر



$x^2+y^2=2y$

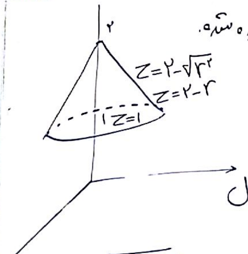
$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

$$\iint_{z=0} F\cdot n\, d\sigma = \iint \left\langle yz, 0, z^2\right\rangle\cdot\left\langle 0,0,1\right\rangle dA$$

$$= \iint_{z=0} 0+0+z^2\, dA = 0$$

مطلوبست شار نیروی $F = x\vec{i} + y\vec{j} + z\vec{k}$ روی سطح $z = 2 - \sqrt{x^2+y^2}$ که توسط رویه $z = 1$ بریده شده.



راه اول

$$\iiint M_x + N_y + P_z \, dV = \iiint 1+1+1 \, dV \;\xrightarrow[\text{پس استوانه‌ای می‌شود}]{\text{چون تقارن داره ۳}}\; 3\int_0^{2\pi}\int_0^1\int_1^{2-r} 1\, r\, dz\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^1 r(2-r) - r \, dr\, d\theta = 3\int_0^{2\pi} \frac{1}{2} - \frac{1}{3}\, d\theta = 3\cdot\frac{1}{6}\theta = \frac{3}{6}\, 2\pi = \pi$$

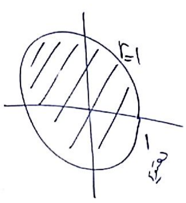
$$\left(\frac{2r^2}{2} - \frac{r^3}{3} - \frac{r^2}{2}\right)$$

راه دوم

$$Z = 1 = 2 - \sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = 2-1$$

$$x^2+y^2 = 1 \quad (r^2 = 1 \longrightarrow r = 1$$



$$\iint_{Z=2-\sqrt{x^2+y^2}} F\cdot n \, d\sigma = \iint \langle x, y, z\rangle \cdot \left\langle -\frac{2x}{2\sqrt{x^2+y^2}}, -\frac{2y}{2\sqrt{x^2+y^2}}, 1 \right\rangle dA$$

$$= \iint \frac{-x^2}{\sqrt{x^2+y^2}} + \frac{-y^2}{\sqrt{x^2+y^2}} + (z)\, dA \;\overset{\text{قطبی}}{=}\; \int_0^{2\pi}\int_0^1 \left(\frac{-r^2\cos^2\theta}{r} + \frac{-r^2\sin^2\theta}{r} + 2 - r\right) r \, dr\, d\theta$$

$$(z = 2-\sqrt{x^2+y^2})$$

$$-\frac{r^3}{3}\cos^2\theta - \frac{r^3}{3}\sin^2\theta + \frac{2r^2}{2} - \frac{r^3}{3}$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos^2\theta - \frac{1}{3}\sin^2\theta + 1 - \frac{1}{3}\, d\theta = \int 1 - \frac{2}{3}\, d\theta = \frac{1}{3}\int_0^{2\pi} d\theta = \frac{2\pi}{3}$$

$$\left(-\frac{1}{3}\right)$$

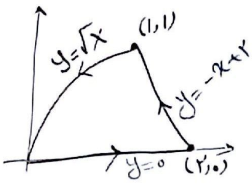
$$\iint F\cdot n\, d\sigma = \iint \langle x, y, z\rangle \cdot \langle 0, 0, 1\rangle$$

$$Z = 1 \quad = \iint_{Z=1} 0 + 0 + z \, dA = \iint 1\, dA$$

$$= R\,\text{(دایره)} = \pi r^2 = \pi.$$

درستی ق گرین

$\vec{F} = (x - y^2)\vec{i} + x\vec{j}$ در منحنی روبرو



$$\oint_C (x-y^2)\,dx + x\,dy = \iint 1+2y \, dA$$

معادله خط گذرا (1,1) (2,0)

$$\frac{y-0}{x-2} = \frac{1-0}{1-2}$$

$$y = -x+2$$

① $\int_{\langle t, 0\rangle} (x-y^2)\,dx + x\,dy = \int_0^2 (t-0)\times 1 + 0 \, dt = \frac{t^2}{2} = \frac{4}{2} - 0 = 2$

② $\int_{\langle t, -t+2\rangle} (x-y^2)\,dx + x\,dy = \int_2^1 \big(t - (t+2)^2\big)\times 1 - t\times 1 \, dt = \int_2^1 4t - t^2 - 4 \, dt = 2t^2 - \frac{t^3}{3} - 4t\Big]_2^1 = \left(2 - \frac{1}{3} - 4\right) - \left(8 - \frac{8}{3} - 8\right) = -2 + \frac{7}{3} = \frac{1}{3}$

($t^2 - 4t + 4$)

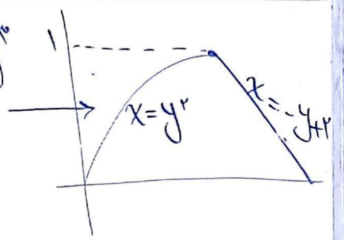
③ $\int_{\langle t^2, t\rangle} (x-y^2)\,dx + x\,dy = \int_1^0 (t^2 - t^2)\times 1 + t^2 \, dt = \frac{t^3}{3}\Big]_1^0 = 0 - \frac{1}{3}$

$$① + ② + ③ = 2 + \frac{1}{3} - \frac{1}{3} = 2$$

$$\int_0^1 \int_{y^2}^{-y+2} (1+2y)\, dx\, dy = \int_0^1 (1+2y)(-y+2) - (1+2y)y^2 \, dy = \int_0^1 -y + 2 - 2y^2 + 4y - y^2 - 2y^3$$

($(1+2y)x$)

$$= \frac{3y^2}{2} - \frac{3y^3}{3} - \frac{2y^4}{4} + 2y\Big]_0^1 = \frac{3}{2} - 1 - \frac{1}{2} + 2 = \frac{2}{2} - \frac{1}{2} + 2 = 2$$



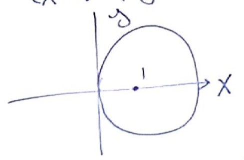
حجم ناحیه محدود به استوانه $x^2+y^2=2x$ و مخروط $z=\sqrt{x^2+y^2}$ و صفحه xy را بیابید.

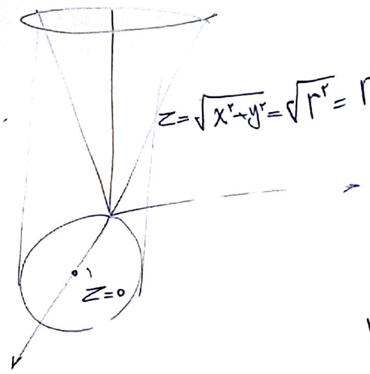
$$\iiint 1\, dV$$

$$x^2+y^2-2x=0$$

$$x^2-2x+1-1+y^2=0$$

$$(x-1)^2+y^2=1$$





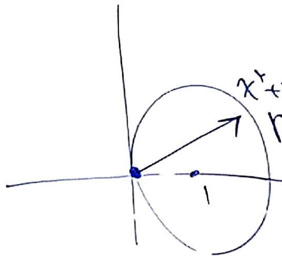
$$z=\sqrt{x^2+y^2}=\sqrt{r^2}=r$$

استوانه‌ای

$$\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{r} 1\; r\,dz\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta=\int_{-\pi/2}^{\pi/2}\frac{8}{3}\cos^3\theta\,d\theta$$

زوج

$$\cos^3\theta=(1-\sin^2\theta)\cos\theta$$



$$x^2+y^2=2x$$

$$r^2=2r\cos\theta$$

$$r=2\cos\theta$$

$$=\frac{16}{3}\int(1-t^2)\,dt=t-\frac{t^3}{3}=\left[\frac{16}{3}\left(\sin\theta-\frac{\sin^3\theta}{3}\right)\right]_0^{\pi/2}=\frac{16}{3}\left[1-\frac{1}{3}\right]=\frac{32}{9}$$